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ON DIFFERENTIAL EVOLUTION SYSTEMS

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Problems of control with incomplete information either on the object parameters or on its current state or on the disturbances occuring are analyzed. Solutions involving an approximate model which can be investigated by the methods of position differential games [1] are described.

1. Many controlled systems with an object describable by differential equations can be represented in the following manner. The current state at instant t is characterized by a position $y[t]$, where $y = y[t]$ is an element of a Hilbert space Y. A family $\{y[\cdot] \mid y[t_*], \Pi[t_*, t^*)\}$ of possible motions $y[\cdot] = \{y[t], t_* \leqslant t \leqslant t^*\}$ is defined for any possible given t_*, $y[t_*]$, t^* and for our action $\Pi[t_*, t^*)$. The real motion $y[\cdot]$ can be interpreted as a concrete choice from

$$y[\cdot] \in \{y[\cdot] \mid y \mid [t_*], \Pi[t_*, t^*)\} \tag{1.1}$$

A game situation is obtained. We — the first player — select $\Pi[t_*, t^*)$; the second player, imaginary in general, selects a realization $y[\cdot]$ in accord with (1.1). The control is examined on the interval $t_0 \leqslant t \leqslant \vartheta$. The function $\Pi[\tau_i, \tau_{i+1}) = U(\tau_i, y[\tau_i], \tau_{i+1}, \varepsilon)$, $\tau_{i+1} > \tau_i$, $\varepsilon > 0$ is called strategy U. Every function $y[t]$, $t_0 \leqslant t \leqslant \vartheta$, $y[t_0] = y_0$ which can be realized by successive choices (1.1), where $t_* = \tau_i$, $t^* = \tau_{i+1}$, $i = 0, 1, 2, \ldots, n$, $\tau_0 = t_0$, $\tau_n = \vartheta$, $\Pi[\tau_i, \tau_{i+1}) = U(\ldots)$, is called a motion $y[t] = y[t, t_0, y_0, U]$.

Let there be given the sets

$$M = [\{t, y\} : t_0 \leqslant t \leqslant \vartheta, \ y \in M(t)] \tag{1.2}$$

$$N = [\{t, y\} : t_0 \leqslant t \leqslant \vartheta, \ y \in N(t)] \tag{1.3}$$

Problem 1.1 For given $\{t_0, y_0\}$, $\vartheta > t_0$, M and N find a strategy U which for every $\varepsilon > 0$ ensures the condition

$$y[\tau] \in M^\varepsilon(\tau), \quad y[t] \in N^\varepsilon(t), \quad t_0 \leqslant t \leqslant \tau \leqslant \vartheta \tag{1.4}$$

for every motion $y[t] = y[t, t_0, y_0, U]$ for at least one $\tau \leqslant \vartheta$ if only $\tau_{i+1} - \tau_i \leqslant \delta(\varepsilon)$.

Here $M^\varepsilon(t)$ and $N^\varepsilon(t)$ are the ε-neighborhoods of $M(t)$ and $N(t)$ in Y. Strategy U can be defined more broadly by including among the arguments another auxiliary variable $w[\tau_i]$ formable in the control loop.

Example 1.1. Let the object be a homogeneous heat-conduction rod $(-\infty < \xi < \infty)$ controlled by a heat source at point $\xi = 0$. A disturbance $v[t]$ is imposed on the source's controlling intensity $u[t]$, $|u| \leqslant \mu$, $|v| \leqslant \nu$. Here $y[t] =$

$\{\zeta(t, \xi), -\infty < \xi < \infty\}$, where $\zeta(t, \xi)$ is the temperature distribution along the rod at instant t. For fixed t the function $\zeta(t, \xi) = \eta(\xi)$ can be treated as an element of the space Y of square-integrable functions $\eta(\xi)$, $-\infty < \xi < \infty$. Here

$$\| y[t] \| = \| \zeta(t, \cdot) \| = \left[\int_{-\infty}^{\infty} \zeta^2(t, \xi)\, d\xi \right]^{1/2} \tag{1.5}$$

The equation of motion is

$$\frac{\partial \zeta}{\partial t} = a^2 \frac{\partial^2 \zeta}{\partial \xi^2} + (u + v)\, \delta(\xi) \tag{1.6}$$

where $\delta(\xi)$ is a δ-function. Let the admissible action $\Pi[t_*, t^*)$ be the constant function $u[t] = u^*$, $t_* \leqslant t < t^*$, $|u^*| \leqslant \mu$. As a disturbance $v[t]$, $t_* \leqslant t < t^*$, we admit any piecewise-continuous function $|v[t]| \leqslant \nu$. Family (1.1) is a set of solutions $y[t] = \{\zeta(t, \xi), -\infty < \xi < \infty\}$, $t_* \leqslant t \leqslant t^*$, of Eq. (1.6) with a known boundary condition $y[t_*] = \{\zeta(t_*, \xi), -\infty < \xi < \infty\}$, when $v[t]$ ranges over all possible disturbances. The choice of $y[\cdot]$ in (1.1) is dictated by the choice of $v[t]$, $t_* \leqslant t < t^*$. The roles of sets M and N can be played, for instance, by the sets

$$M = [\{t, \zeta(t, \cdot)\} : t_0 \leqslant t \leqslant \vartheta, \ \| \zeta(t, \cdot) \| \leqslant \alpha]$$

$$N = [\{t, \zeta(t, \cdot)\} : t_0 \leqslant t \leqslant \vartheta, \ \| \zeta(t, \cdot) \| \leqslant \beta]$$

2. Together with the original y-system (1.1) we examine a certain w-model of it, whose current state — the positions $w[t]$ — are elements of space Y. We assume that the variation of $w[t]$, $w[t_*] = w_*$ on the interval $t_* \leqslant t \leqslant t^*$ is determined by the forces $F^{(1)}[t_*, t^*)$ and $F^{(2)}[t_*, t^*)$, so that $w[t] = w[t, t_*, w_*, F^{(1)}, F^{(2)}]$. Here $F^{(2)}$ is chosen first and then $F^{(1)}$. The following method of forming $w[t]$, $t \geqslant t_0$, from the position $w[t_0] = w_0$ at the expense of successive choice of forces $F^{(2)}[\tau_i^*, \tau_{i+1}^*)$ $(\tau_0^* = t_0,\ i = 0, 1, 2, \ldots)$ is termed procedure Q. At an instant $\tau_i^* < \vartheta$ the procedure Q fixes the next half-open interval $\tau_i^* \leqslant t < \tau_{i+1}^*$ and the force $F^{(2)}[\tau_i^*, \tau_{i+1}^*)$ as functions of the history $\{w[t], t_0 \leqslant t \leqslant \tau_i^*\}$. Force $F^{(1)}[\tau_i^*, \tau_{i+1}^*)$ may be arbitrary from a set of admissible ones. The set of admissible procedures Q is restricted only by the condition that procedure Q must not generate a realization $w[t]$ with an infinite number of partitions τ_i^* on the interval $[t_0, \vartheta]$ for any sequence of choices of $F^{(1)}[\tau_i^*, \tau_{i+1}^*)$.

The following statement is valid.

Lemma 2.1. For every choice of $\{t_*, w_*\}$, $\vartheta \geqslant t_*$, M_*, N_* and $t^* \in [t_*, \vartheta]$ one and only one of the following two statements is valid for the initial position $w[t_*] = w_*$: 1) a procedure Q exists which excludes the fulfillment of the condition

$$w[\tau] \in M_*(\tau), \quad w[t] \in N_*(t), \quad t_* \leqslant t \leqslant \tau \leqslant \vartheta \tag{2.1}$$

for all the motions $w[t] = w[t, t_*, w_*, Q]$ generated by it; 2) for any choice of force $F^{(2)}\ [t_*,\ t^*)$ we can find a force $F^{(1)}\ [t_*,\ t^*)$ which for the motion $w\ [t] = w[t,\ t_*,\ w_*,\ F^{(1)},\ F^{(2)}]$ either ensures the fulfillment of the inclusion

$$w\ [\tau] \in M_*\ (\tau), \quad w\ [t] \in N_*\ (t), \quad t_* \leqslant t \leqslant \tau \leqslant t^* \tag{2.2}$$

for some τ or realizes a position $w\ [t^*] = w^*$ for which, as for the initial position, there does not exist a procedure Q excluding the condition

$$w\ [\tau] \in M_*\ (\tau), \quad w\ [t] \in N_*\ (t), \quad t_* \leqslant t \leqslant \tau \leqslant t^* \tag{2.3}$$

for all motions $w\ [t] = w\ [t,\ t^*,\ w^*,\ Q]$.

3. We say that the w-model approximates the y-system from below if for every possible pair $y\ [t_*] = y_*,\ w\ [t_*] = w_*, \| y_* - w_* \| < \varepsilon_0$ we can find $\Pi\ [t_*,\ t^*)$ for every $t^* \in [t_*, \vartheta],\ t^* - t_* \leqslant \delta$, such that for every realization $y\ [\cdot]$ from (1.1) we can find $F^{(2)}\ [t_*,\ t^*)$ so that for any $F^{(1)}\ [t_*,\ t^*)$ the inequality

$$\|\ y\ [t] - w\ [t]\ \|^2 \leqslant (1 + \varkappa\ \ (t - t_*))\ \|\ y_* - w_*\ \|^2 + \varphi\ (\delta)\ (t - t_*) \tag{3.1}$$

is valid for the corresponding motions $y\ [t]$ and $w\ [t]$ for $t_* \leqslant t \leqslant t^*$ where $\varkappa$ is a constant and $\lim\ \varphi\ (\delta) = 0$ as $\delta \to 0$. We can have another sequence of choice of forces: we find $F^{(2)}$ so that for every $F^{(1)}$ we can find Π such that for any realization $y\ [\cdot]$ from (1.1) estimate (3.1) is valid for the motions $y\ [t]$ and $w\ [t]$ The following assertion, provable by a well-known scheme (see [1], pp. 59-70), is valid.

Lemma 3.1. Let the w-model approximate the y-system from below. If for the position $y\ [t_0] = y_0 = w\ [t_0] = w_0$ there does not exist a procedure Q excluding (2.1) for $M_*\ (t) = M^\alpha\ (t)$ and $N_*\ (t) = N^\alpha\ (t)$ for any $\alpha > 0$, then we can find a strategy $U\ (\tau_i,\ y\ [\tau_i],\ \ \tau_{i+1},\ \ \varepsilon)$ solving Problem 1.1.

Suppose that for every $\tau \in (t_*,\ t^*)$ the forces $F^{(i)}\ [t_*,\ t^*)$ determine the forces $F^{(i)}\ [t_*, \tau)$ and $F^{(i)}\ [\tau,\ t^*)$ and, conversely, the forces $F^{(i)}\ [t_*,\ \tau)$ and $F^{(i)}\ [\tau,\ t^*)$ determine the forces $F^{(i)}\ [t_*,\ t^*)\ (i = 1{,}2)$. In this case the corresponding pieces of motions $w\ [t],\ t_* \leqslant t \leqslant \tau$ and $w\ [t],\ \tau \leqslant t \leqslant t^*$, join together in a natural way. Let us assume that $y\ [\cdot]$ is selected from (1.1) as the action of the operator

$$y\ [\cdot] = Y\ \{y\ [t_*],\ \Pi\ [t_*,\ t^*)\} \tag{3.2}$$

prescribed a priori. The class Y of admissible operators (3.2) can be pre-specified if the process is regarded from the second player's positions.

We say that the w-model approximates the y-system from above if for every possible pair $y\ [t_*] = y_*,\ w\ [t_*] = w_*$ we can find the Y in (3.2) for every $t^* \in [t_*,\ t_* + \delta \leqslant \vartheta]$, so that for every $\Pi\ [t_*,\ t^*)$ we can find for every set $\{F^{(2)}\ [\tau_i{}^*,\ \tau_{i+1}^*),\ i = 0,1,\ldots,n;\ \tau_0{}^* = t_*,\ \ \tau_n{}^* = t^*\}$ a set $\{F^{(1)}\ [\tau_i{}^*,\ \tau_{i+1}^*)\}$ which ensures inequality (3.1). Here $F^{(1)}\ [\tau_i^*,\ \tau_{i+1}^*)$ are chosen after $F^{(2)}\ [\tau_i{}^*,\ \tau_{i+1}^*)$ while τ_{i+2}^* and $F^{(2)}\ [\tau_{i+1}^*,\ \tau_{i+2}^*)$ are chosen after $F^{(1)}\ [\tau_i{}^*,\ \tau_{i+1}^*)$. We can have another sequence of choices of forces: we can find a method for selecting the set $\{F^{(1)}\ [\tau_i{}^*,\ \tau_{i+1}^*)\}$ such that for every set $\{F^{(2)}\ [\tau_i{}^*,\ \tau_{i+1}^*)\}$ we can find the Y in (3.2) so that for every $\Pi\ [\tau_i,\ \ \tau_{i+1})$ the estimate will be valid for motions $y\ [t]$ and $w\ [t]$. The following statement, which can once again be proved by a well-known scheme (see [1], pp. 59-70), is valid.

Theorem 3.1. Let the y-system be regular, i.e., let it admit of a w-model which approximates it from both below and above. Then for every choice of $\{t_0, y_0\}$, $\vartheta \geqslant t_0$, M and N one and only one of the following assertions is valid for the initial position $y[t_0] = y_0$:

1) a strategy $U(\tau_i, y[\tau_i], \tau_{i+1}, \varepsilon)$ exists solving Problem 1.1;

2) there exist $\gamma > 0$ and a choice of operators Y of (3.2) ($t_* = \tau_i$, $t^* = \tau_{i+1}$) as a function of $\{\tau_i, y[\tau_i]\}$, which for every motion $y[t]$ excludes (1.4) with $\varepsilon = \gamma$, if only $\tau_{i+1} - \tau_i \leqslant \delta_*(\gamma)$.

This statement corresponds to the theorems on the alternative in [1].

4. It is difficult to satisfy condition (3.1) if space Y is infinite-dimensional. In such cases it is appropriate that the w-model approximate not the y-system itself but some approximation or some mapping of it. Let σ be a certain parameter. We introduce a transformation

$$y_\sigma[t] = f_\sigma(t, y[t]) \tag{4.1}$$

where the $y_\sigma[t]$ are elements of a Hilbert space Y_σ. With the sets $M(t)$ and $N(t)$ of Problem 1.1 we associate sets $M_\sigma(t)$ and $N_\sigma(t)$. Let Σ_α, $\alpha > 0$, be sets of values of σ, satisfying the conditions $\Sigma_\alpha \subset \Sigma_\beta$ when $\beta > \alpha$. Suppose that the conditions are fulfilled. From $y_\sigma[t] \in M_\sigma^\varepsilon(t)$, $\varepsilon > 0$ follows $y[t] \in M^{\gamma(\sigma,\varepsilon)}(t)$, and for any $\alpha > 0$ we can find $\Sigma_{\xi(\alpha)}$ and $\varepsilon(\alpha) > 0$, so that $\gamma(\sigma, \varepsilon) < \alpha$ for $\sigma \in \Sigma_{\xi(\alpha)}$ and $\varepsilon \leqslant \varepsilon(\alpha)$. For each $\beta > 0$ we can find $\Sigma_{\xi(\beta)}$ and $(\beta) > 0$ so that from $y_\sigma[t] \notin M_\sigma^\beta(t)$ follows $y[t] \notin M^{\varepsilon(\beta)}$ when $\sigma \in \Sigma_{\xi(\beta)}$. Let the very same conditions be satisfied for $N(t)$ and $N_\sigma(t)$.

Assume that the systems

$$y_\sigma[\cdot] \in \{y_\sigma[\cdot] \mid y_\sigma[t_*], \Pi[t_*, t^*)\} \tag{4.2}$$

into which system (1.1) can be induced are regular, i.e., admit of w_σ-models approximating them from below and above. Then Theorem 3.1 is again true for such a y-system (1.1) regular in approximation.

The system in Example 1.1 is regular in approximation. Indeed, over the variable $y[t] = \{\zeta(t, \xi), -\infty < \xi < \infty\}$ we perform the transformation

$$\zeta_\sigma(t,\xi) = \int_{-\infty}^{\infty} \frac{1}{2a\sqrt{\pi(\vartheta+\sigma-t)}} \exp\left[-\frac{(\xi-\eta)^2}{4a^2(\vartheta+\sigma-t)}\right] \zeta(t,\eta)\, d\eta \tag{4.3}$$

where $\sigma > 0$ is any one fixed value. System (4.2) is determined by the equality

$$\zeta_\sigma(t,\xi) = \zeta_\sigma(t_*,\xi) + \int_{t_*}^{t} \frac{1}{2a\sqrt{\pi(\vartheta+\sigma-\tau)}} \exp\left[-\frac{\xi^2}{4a^2(\vartheta+\sigma-\tau)}\right] (u^*[\tau] + v[\tau])\, d\tau \tag{4.4}$$

where for a fixed action $\Pi[t_*, t^*) \div u^*[t]$, $t_* \leqslant t < t^*$, we obtain the whole family $\{\zeta_\sigma(t, \xi), t_* \leqslant t \leqslant t^*, -\infty < \xi < \infty\}$ when $v[t], t_* \leqslant t \leqslant t^*$, ranges over all possible realizations $v[t]$. In accord with (1.7) and (1.8) the sets $M_\sigma(t)$ and $N_\sigma(t)$ are defined as sets of the functions $\zeta_\sigma(t, \xi)$ of (4.3) wherein the functions $\zeta(t, \eta)$ satisfy the conditions $\|\zeta(t, \cdot)\| \leqslant \alpha$ and $\|\zeta(t, \cdot)\| \leqslant \beta$. All functions $\zeta(t, \xi)$, being solutions of Eq. (1.6) and corresponding to all possible $u[t]$ and $v[t]$, $t_0 \leqslant t \leqslant \vartheta$ for a fixed initial position $\zeta(t_0, \xi)$, are contained in some compactum in Y. Hence it follows that the necessary relations between M_σ and M and between N_σ and N are fulfilled.

Let us specify the motions $w_\sigma[t] = \{\zeta_\sigma(t, \xi)_w, -\infty < \xi < \infty\}$ for the ω_σ-model by the equation

$$w_\sigma^{\cdot} = \left\{\frac{\partial \zeta_\sigma(t, \xi)_w}{\partial t}, -\infty < \xi < \infty\right\} = \tag{4.5}$$
$$\left\{\frac{1}{2a\sqrt{\pi(\vartheta+\sigma-t)}} \exp\left[-\frac{\xi^2}{4a^2(\vartheta+\sigma-t)}\right], -\infty < \xi < \infty\right\}(u_* + v_*), \quad (|u_*| \leqslant \mu, |v_*| \leqslant \nu)$$

The choice of the measurable function $u_*[t]$ $(t_* \leqslant t < t^*)$ will be the force $F^{(1)}[t_*, t^*)$ and the choice of the measurable function $v_*[t]$ $(t_* \leqslant t < t^*)$ will be the force $F^{(2)}[t_*, t^*)$. Let $\mu > \nu$. Analogously to Problem 1.1 for system (1.1) the problem for w_σ-model (4.5) can be solved according to the material presented (see [1], pp. 207-233). We conclude: Problem 1.1 for system (1.6) has a solution if and only if at least one motion $w^\circ[t] = \zeta^\circ(t, \cdot)$, $w^\circ[t_0] = w_0 = y_0 = \zeta^\circ(t_0, \cdot)$, of the system described by the equation

$$w^{\cdot} = \frac{\partial \zeta}{\partial t} = \left\{\frac{1}{2a\sqrt{\pi(\vartheta+\sigma-t)}} \exp\left[-\frac{\xi^2}{4a^2\sqrt{(\vartheta+\sigma-t)}}\right], -\infty < \xi < \infty\right\} p, \quad |p[t]| \leqslant \mu - \nu$$

satisfies the conditions

$$w^\circ[\tau] \in M_\sigma(\tau), \quad w^\circ[t] \in N_\sigma(t), \quad t_0 \leqslant t \leqslant \tau \leqslant \vartheta$$

The resolving strategy U will be extremal to this stable path $w^\circ[t]$ and the control $u[t] = u^\circ$, $\tau_i \leqslant t < \tau_{i+1}$, will be determined from the condition

$$u^\circ \int_{-\infty}^{\infty} \frac{1}{2a\sqrt{\pi(\vartheta+\sigma-\tau_i)}} \exp\left[-\frac{\xi^2}{4a^2(\vartheta+\sigma-\tau_i)}\right] \times (\zeta_\sigma(\tau_i, \xi) - \zeta^\circ(\tau_i, \xi))\, d\xi = \min_{|u| \leqslant \mu}$$

The right-hand sides of Eq.(4.5) belong to some compactum in Y. Therefore, the w_σ-model can be approximated by finite-dimensional $w_\sigma^{(n)}$ models.

5. Let us consider the question of approximating a w-model by a finite-dimensional $w^{(n)}$-model. Let $w^{(n)}$ be an element of some n-dimensional subspace $Y^{(n)}$ of space Y. We say that a sequence of $w^{(n)}$-models approximates a w-model from below (above) if for every $F^{(2)}[t_*, t^*)$ $(F^{(1)}[t_*, t^*))$ we can find $F_n^{(2)}[t_*, t^*)$ $(F_n^{(1)}[t_*, t^*))$ so that for every $F_n^{(1)}[t_*, t^*)(F_n^{(2)}[t_*, t^*))$ we can find $F^{(1)}[t_*, t^*)$ $(F^{(2)}[t_*, t^*))$ so that the estimate

$$\| w[t] - w^{(n)}[t] \|^2 \leqslant (1 + \varkappa_n (t - t_*)) \| w_* - w_*^{(n)} \|^2 + \varphi(\delta, n)(t - t_*) \tag{5.1}$$

is valid when $t_* \leqslant t \leqslant t^* \leqslant \delta$, where $\lim \varphi(\delta, n) = 0$ as $n \to \infty$ and $\delta \to 0$. Let the y-system (1.1) be regular in approximation and let each w_σ-model in its own turn be approximated from above and below by a sequence of $w_\sigma^{(n)}$-models. Then the solution of Problem 1.1 for the given y-system is determined by the solutions of the appropriate analogous problems for the $w_\sigma^{(n)}$-models with appropriate σ and large n. The control process can in fact be effected by realizing in the regulation scheme the cascade: $\{y$-system, y_σ-system, w_σ-model, $w_\sigma^{(n)}$-model$\}$. In this case the controls Π, $F_\sigma^{(2)}$, $F_{\sigma,n}^{(2)}$, $F_{\sigma,n}^{(1)}$, $F_\sigma^{(1)}$ will be selected from conditions (3.1) and (5.1) and from the solvability conditions for the analog of Problem 1.1 for the $w_\sigma^{(n)}$-model. The solution of this analog of Problem 1.1 can be constructed by one of the methods in [1].

6. It is well known that the solution of control game problems for finite-dimensional systems can be regularized in a number of cases by superposing small random disturbances. When a $w_\sigma^{(n)}$-model is introduced in the control loop, this regularization can be given a real meaning.

Let a certain system be described by the equation

$$x^{\cdot} = f(t, x, u, v), \quad u \in P, \quad v \in Q \tag{6.1}$$

where x is an n-dimensional vector, P and Q are compacta, function f is continuous and satisfies the Lipschitz condition in x and the condition $\| f \| \leqslant \varkappa (1 + \| x \|)$, $\varkappa = \text{const}$. Let us consider as well the corresponding stochastic system

$$dx = f(t, x, u, v)\, dt + \alpha\, dz[t] \tag{6.2}$$

where $z[t]$ is a standard nondegenerate Wiener process [2] and α is a small parameter. Let closed sets M and N be given in space $\{t, x\}$. For system (6.1) one and only one of the following assertions (see [1], pp. 353-371) is valid for every initial position $x[t_0] = x_0$:

1) a counter-strategy $V \div v(t, x, u)$ and numbers $\varepsilon = \varepsilon_0 > 0$ and $\delta > 0$ can be found such that for every solution $x_\Delta[t] = x_\Delta[t, t_0, x_0, V]$ of the equation

$$x_\Delta^{\cdot} = f[t, x_\Delta[t], u[t], v(\tau_i, x_\Delta[\tau_i], u[t]))$$
$$\tau_i \leqslant t < \tau_{i+1}, \quad \tau_0 = t_0, \quad \tau_m = \vartheta \tag{6.3}$$

the condition

$$x_\Delta[\tau] \in M^\varepsilon(\tau), \quad x_\Delta[t] \in N^\varepsilon(t), \quad t_0 \leqslant t \leqslant \tau \leqslant \vartheta \tag{6.4}$$

with $\varepsilon = \varepsilon_0$ is excluded if only $\tau_{i+1} - \tau_i \leqslant \delta$;

2) a strategy $U \div u(t, x)$ can be found such that for every choice of $\varepsilon > 0$ we can find $\delta(\varepsilon) > 0$ so that condition (6.4) is fulfilled for every solution $x_\Delta[t] = x_\Delta[t, t_0, x_0, U]$ of the equation

$$x_\Delta^{\boldsymbol{\cdot}} = f(t, x_\Delta[t], \quad u(\tau_i, x_\Delta[\tau_i]), v[t]) \qquad (6.5)$$
$$\tau_i \leqslant t < \tau_{i+1}$$

if only $\tau_{i+1} - \tau_i \leqslant \delta(\varepsilon)$

Let us consider the motions $x[t] = x[t, t_*, x_*, u_*, v[\cdot]]$ of (6.1) and $x^{(\alpha)}[t] = x[t, t_*, x^*, u[\cdot], v^*(u[\cdot])]$ of (6.2) for $t_* \leqslant t \leqslant t^*$ where the controls u_* and $v^*(u)$ are chosen from the conditions

$$\langle (x_* - x^*)\cdot f(t_*, x_*, u_*, v)\rangle = \min_u \max_v \qquad (6.6)$$
$$\langle (x_* - x^*)\cdot f(t_*, x^*, u, v^*)\rangle = \max_v \qquad (6.7)$$

Here $\langle x\cdot f\rangle$ is the scalar product. The function $v[\cdot] = \{v[t], t_* \leqslant t < t^*\}$ in $x[t]$ is any function Lebesgue-measurable in t. The motion $x^{(\alpha)}[t]$ is a probabilistic diffusion process. The function $u[\cdot]$ in $x^{(\alpha)}[t]$ can be arbitrary, including random, and, possibly, can be connected with the function $x^{(\alpha)}[\cdot]$ for which the solutions $x^{(\alpha)}[t]$ of (6.2) can be formalized in the standard concepts of the theory of diffusion processes under the condition that $v(u)$ is a Borel-measurable function. The estimate

$$M\{\| x[t] - x^{(\alpha)}[t]\|^2\} \leqslant (1 + \varkappa(t - t_0))\| x_* - x^*\|^2 + \varphi(\alpha, \delta)(t - t_*) \qquad (6.8)$$

with $t - t_* \leqslant \delta$ where $\lim \varphi(\alpha, \delta) = 0$ as $\{\alpha, \delta\} \to 0$ and $M\{\xi\}$ is the mean, is valid. The estimate (6.8) is valid also for the motions $x[t] = x[t, t_*, x_*, u[\cdot], v_*(u[\cdot])]$ of (6.1) and $x^{(\alpha)}[t] = x[t, t_*, x^*, u^*, v[\cdot]]$ of (6.2) for $t_* \leqslant t < t^*$ where the controls $v_*(u)$ and u^* are chosen from the conditions

$$\langle (x^* - x_*)\cdot f(t_*, x_*, u, v_*)\rangle = \max_v' \qquad (6.9)$$
$$\langle (x^* - x_*)\cdot f(t_*, x_*, u^*, v)\rangle = \min_u \max_v \qquad (6.10)$$

The function $u[\cdot] = \{u[t], t_* \leqslant t < t^*\}$ in $x[t]$ is any function Lebesgue-measurable in t and $v_*(u)$ is Borel-measurable. For $x^{(\alpha)}[t]$ the function $v[\cdot]$ can be random and, in addition, connected with $x^{(\alpha)}[t]$.

The following assertions are obtained from the estimates given by well-known arguments (see [1], pp. 329-347). For the initial position $x[t_0] = x_0$ let there exist a strategy $U \div u(t, x)$ which ensures condition (6.4) for the motions $x_\Delta[t]$ of (6.5). Then for any $\varepsilon > 0$ and $p < 1$ we can find $\alpha(\varepsilon, p) > 0$ for system (6.2) with the same initial position $x[t_0] = x_0$, such that no formalizable choice of control v in (6.2) can exist which would exclude the conditions

$$x^{(\alpha)}[\tau] \in M^\varepsilon(\tau), \quad x^{(\alpha)}[t] \in N^\varepsilon(t), \quad t_0 \leqslant t \leqslant \tau \leqslant \vartheta \qquad (6.11)$$

with a probability greater than $1-p$ if only $\alpha \leqslant \alpha(\varepsilon, p)$ in (6.2). In particular, then a counter-strategy $V \div v(t, x, u)$ cannot exist for system (6.2), which would exclude conditions (6.11) with a probability greater than $1-p$. Conversely, let a counter-strategy $V \div v(t, x, u)$ exist for the given initial position $x[t_0] = x_0$, which excludes for motions $x_\Delta[t]$ of (6.3) the condition (6.4) with some value of $\varepsilon = \varepsilon_0 > 0$ if only $\tau_{i+1} - \tau_i \leqslant \delta$. Then for any $\varepsilon < \varepsilon_0$ and $p < 1$ we can find $\alpha(\varepsilon, p) > 0$ for system (6.2) with the same initial position $x[t_0] = x_0$, such that no formalizable position choice of control u in (6.2) can exist which would ensure conditions (6.11) with a probability greater than $1-p$ if $\alpha \leqslant \alpha(\varepsilon, p)$ in (6.2). In particular, a strategy $U \div u(t, x)$ cannot exist for system (6.2), which would ensure conditions (6.11) with a probability greater than $1-p$.

If the saddle-point condition for the small game (see [1], pp. 55-57) is fulfilled, then it is sufficient to restrict ourselves only to the strategies $U \div u(t, x)$ and $V \div v(t, x)$ in the preceding assertions.

It is well known that for system (6.2) the solution of encounter-evasion problems is, in general, more regular than for system (6.1). In particular, in many cases of control game problems for system (6.2) there exist sufficiently smooth solutions of the dynamic programing equation which then is a quasi-linear parabolic partial differential equation. From the assertions in Sect. 6 it follows that in such cases it is advisable to include in the control cascade, after the $w_\sigma^{(n)}$-model described by Eq. (6.1), a further stochastic $w_\sigma^{(n)*}$-model described by Eq. (6.2) with a sufficiently small value of $\alpha > 0$. Here the role of the first leader (see [1], pp. 248-254) is played by the motion $w_\sigma^{(n)*}[t]$ which through the cascade $\{y[t], y_\sigma[t], w_\sigma[t], w_\sigma^{(n)}[t], w_\sigma^{(n)*}[t]\}$ takes the motion $y[t]$ to the target needed with a probability arbitrarily close to unity.

7. In accord with the facts discussed in Sec. 6 the advisability of including the stochastic leader $w_\sigma^{(n)*}[t]$ in the control loop is connected with the fact that for a wide class of differential games for system (6.1), as a consequence of estimates of form (6.8), the value of the corresponding stochastic differential game for system (6.2) converges as $\alpha \to 0$ to the value of the original game for system (6.1).

For example, suppose that for the original y-system (1.1) we are to choose the optimal control $\Pi[\tau_i, \tau_{i+1}) = U(\tau_i, y[\tau_i], \tau_{i+1}, \ldots)$ which ensures $\min_\Pi \max_{y[\cdot]} \omega^*(y[\vartheta])$ where $\omega^*(y)$ is a given function. The problem mentioned corresponds to a Problem 1.1 wherein

$$M_c = [\{t, y\},\ t = \vartheta,\ \omega^*(y) \leqslant c], \quad N_c = [\{t, y\},\ t_0 \leqslant t \leqslant \vartheta,\ y \in Y]$$

and we are required to solve the Problem 1.1 with the smallest value $c = c_0$ for which it has a solution. For this original problem suppose that at the stage of the $w_\sigma^{(n)}$-model we are dealing with a differential game (see [1], pp. 71-79) for system (6.1) with the index

$$\gamma = \min_U \max_V \omega(x[\vartheta]), \quad U \div u(t, x),\ V \div v(t, x, v) \tag{7.1}$$

This game has the saddle point $\{U^\circ, V^\circ\}$ which corresponds to the game value γ°

(see [1], pp. 71-79). If the saddle-point condition for the small game (see [1], pp. 55-57) is fulfilled, then the game for system (6.1) with index (7.1) has a saddle point at the pair of strategies $U^\circ \div u^\circ (t, x)$, $V^\circ \div v^\circ (t, x)$. With this game we can associate a stochastic game for system (6.2) with the index

$$\gamma_\alpha = \min_U \max_V M \{\omega (x^{(\alpha)} [\vartheta])\} \tag{7.2}$$

We assume that function $\omega (x)$ is bounded for $-\infty < x < \infty$ and for sufficiently smooth $\gamma_\alpha (t_0, x_0)$. Then the solution of the given game is determined by the smooth solution of the dynamic programing equation with boundary condition

$$\frac{\partial \gamma_\alpha}{\partial t} + \frac{\alpha^2}{2} \sum_{i=1}^{n} \frac{\partial^2 \gamma_\alpha}{\partial x_i^2} + \min_{u \in P} \max_{v \in Q} \left[\sum_{i=1}^{n} \frac{\partial \gamma}{\partial x_i} f_i (t, x, u, v) \right] = 0$$

$$\gamma_\alpha (\vartheta, x) = \omega (x)$$

According to [3], such a solution $\gamma_\alpha (t, x)$ exists. The controls u and v are determined from the corresponding minimax conditions as Borel-measurable functions $u^* (t, x)$ and $v^*(t, x, u)$. Hence, according to [2], follows the existence of the motion $x^{(\alpha)} [t]$ as a solution of the corresponding (weak) diffusion equation (6.2). Therefore, the game admits of a rigorous natural formalization with value (7.2) in the classes of such measurable strategies $u (t, x)$ and $v (t, x, u)$

From estimates (6.6) — (6.10) it follows that $\lim_{\alpha \to 0} \gamma_\alpha = \gamma^\circ$.

REFERENCES

1. Krasovskii, N.N. and Subbotin, A.I., Position Differential Games. Moscow, "Nauka", 1974.
2. Gikhman, I.I. and Skorokhod, A.V., Theory of Random Processes, Vol.3. Moscow, "Nauka", 1975.
3. Ladyzhenskaia, O.A., Solonnikov, V.A. and Ural'tseva, N.N., Linear and Quasilinear Equations of Parabolic Type. Moscow, "Nauka", 1976.

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